

# SPACE SYMMETRIES AND QUANTUM BEHAVIOUR OF FINITE ENERGY CONFIGURATIONS IN $SU(2)$ -GAUGE THEORY

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## Abstract

The quantum properties of localized finite energy solutions to classical Euler-Lagrange equations are investigated using the method of collective coordinates. The perturbation theory in terms of inverse powers of the coupling constant  $g$  is constructed, taking into account the conservation laws of momentum and angular momentum (invariance of the action with respect to the group of motion  $M(3)$  of 3-dimensional Euclidean space) rigorously in every order of perturbation theory.

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## 1. Introduction

The discovery of extended objects in field theory has required the development of methods for describing their quantum properties. One of the most powerful methods is based on introducing collective coordinates, which are related to the symmetry group of the physical system. This method has a long history beginning from 1940 [1] when it was applied to the model of fixed source. In [2] this method has been formulated in the mathematically rigorous and consistent form. In connection with nonlinear classical field theories this method has attracted attention and in the papers of many authors it was developed for interacting systems with arbitrary symmetries and applied to various quantum field theoretical models with either the operator [3] or path integral formulation of the field theory [4]. The general covariant formulation of the method for the multicomponent fields of any kind on the basis of Dirac's generalized Hamiltonian theory [12] in the frame of the path integral approach to the quantum theory is given in [5]. An alternative approach to the covariant description of field theoretical models has been given in [6]. The study of localized solutions in realistic models has shown, that their quantum properties enable us to describe spin structure [7, 8]. In order to achieve this the collective coordinates must be introduced, which are related to the full symmetry of the 3-dimensional space, that is the symmetry with respect to the transformations of the group (named  $M(3)$ ) of motion of 3-dimensional Euclidean space [7, 14].

In this paper we describe the spin structure of pure  $SU(2)$ -gauge theory, which admits classical localized finite energy regular configurations [10] (a review on various possible such configurations is given in [11]). This is the simplest model that can be studied. The more complicated gauge theories require to describe the internal structure also. There is a number of papers, in which quantum properties of such solitonic configurations have been investigated (see [9, 11] and references therein). The important point in the study of gauge theories is, of course, the gauge freedom. One tries always to introduce the gauge conditions in order to cancel the unphysical degrees of freedom and then constructs the quantum theory. In ref. [13] a complete analysis of possible gauges has been performed showing either positive or negative features of these. We only point out that the difficulties regarding the possible gauges are related either to the singularities of gauge transformations or to the nonlocal nature of the reduced phase space and the violation of the spherical symmetry. In the following we proceed from full phase space

in constructing the Hamiltonian picture and pass to the quantum theory as usual. The constraints will then be understood as operator equalities acting on a corresponding Hilbert space. So the problem reduces to the construction of the appropriate space of physical states, that is the space of vectors which are simultaneously the eigenvectors of the Hamilton operator and conserved quantities and fulfills the requirements of the constraints.

## 2. Hamiltonian formalism

We consider nonabelian  $SU(2)$ -gauge theory in 4-dimensional Minkowskian space with the action functional (the metric  $(1, -1, -1, -1)$  will be used)<sup>1</sup>

$$S = \int d^4x \mathcal{L}(A_{\mu a}, \partial_\mu A_{\nu a}) = -\frac{1}{4} \int d^4x F_{\mu\nu a} F^{\mu\nu}{}_a. \quad (1)$$

The field strength tensor is:

$$F_{\mu\nu a} = \partial_\mu A_{\nu a}(x) - \partial_\nu A_{\mu a}(x) + g\varepsilon_{abc}A_{\mu b}(x)A_{\nu c}(x), \quad (2)$$

with  $A_{\mu a}(x)$  as a vector field belonging to the adjoint representation of the gauge group  $SU(2)$  and being a solution of the set of differential equations:

$$\partial^\mu F_{\mu\nu a} + g\varepsilon_{abc}A_b^\mu(x)F_{\mu\nu c} = 0. \quad (3)$$

We suggest that there is a space-localized finite energy static solution. Proceeding from translational invariance of the theory we suppose that the solution depends on parameters  $x_{0i}$ , which represent the position of the localized solution at the initial time. Translations in space-time change the position of the solution in such a way, that it could be useful to parametrize the solutions as  $A_{\mu a}(x) = A_{\mu a}(\mathbf{x}, c_k t + x_{0k})$  which means that  $\partial_0$  has to be replaced by  $c_k \partial_k$ . We see that if  $A_{\mu a}(x) = g^{-1} \tilde{A}_{\mu a}(x)$ , then  $\tilde{A}_{\mu a}(x)$  does not depend on  $g$ . Let us pass to the Hamiltonian formulation of the theory as we are going to work in the Schrödinger picture. The Lagrangian of the model we are considering is singular, that is the canonical variables fulfill some constraint equations. We shall make use of the generalized Hamiltonian formalism developed by Dirac [12] for such physical systems for constructing the Hamilton's equations of motion. Let us define  $A_{\mu a}(x)$  and their conjugate momenta

$$\pi_a^\mu(x) = \frac{\partial \mathcal{L}}{\partial \dot{A}_{\mu a}(x)} = F^{\mu 0}{}_a \quad (4)$$

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<sup>1</sup>We use Greek letters for vectors in Minkowskian space, the letters i,j,k... for vectors in Euclidean space and a,b,c... for vectors in isotopic space.

as the set of the phase space variables. One finds immediately primary constraints:

$$\pi_a^0(x) = 0, \quad a = 1, 2, 3. \quad (5)$$

The canonical Hamiltonian is now:

$$H_c = \int d^3x \left\{ -\frac{1}{2} \pi_a^i(x) \pi_{ia}(x) + \frac{1}{4} F_{ija} F_{ija} - A_{0a}(x) D_i \pi_a^i(x) \right\}, \quad (6)$$

with  $D_\mu$  as covariant derivative:

$$(D_\mu)_{ac} = \delta_{ac} \partial_\mu + \varepsilon_{abc} A_{\mu b}(x) = \delta_{ac} \partial_\mu + (I_b)_{ac} A_{\mu b}(x). \quad (7)$$

The quantities  $(I_b)_{ac}$  are elements of the adjoint representation of the group  $SU(2)$ . There are also secondary constraints in the theory, which are results of the equations of motion, namely the conditions  $\dot{\pi}_a^0(x) = 0$  yield

$$\Phi_a(x) = D_i \pi_a^i(x) = 0. \quad (8)$$

One can check that there are no more constraints in the theory. Using the definition of Poisson brackets without taking into account any constraints gives the following relations:

$$\{\Phi_a(x), \Phi_b(y)\} = g \varepsilon_{abc} \Phi_c(x) \delta^3(x - y), \quad (9)$$

which mean that the constraints  $\Phi_a(x)$  generate local time-independent gauge transformations. From now on we write the Hamiltonian as follows:

$$H = H_0 + \int d^3x v_a(x) \Phi_a(x), \quad (10)$$

in which  $H_0$  is the canonical Hamiltonian and  $v_a(x)$  are Lagrange multipliers. In writing this we assumed the additional conditions (or gauge)  $A_{0a} = 0$ , so that the variables  $A_{0a}$  and  $\pi_a^0(x)$  drop out. In general one introduces the gauge conditions  $h_a = 0$  with the non-vanishing Poisson brackets of these with  $\pi_a^0(x)$  and considers the equation of motion for  $h_a$ . This makes it possible to determine  $A_{0a}$  as a function of the rest of the canonical variables and we again get the Hamiltonian, which does not depend on  $A_{0a}$  and  $\pi_a^0(x)$  [13]. Nevertheless our choice of the gauge does not affect the symmetry properties we want to study. Thus  $\Phi_a(x)$  are the only constraints we have

in the theory. In order to pass to the quantum theory we must choose another gauge condition, define physical phase space and corresponding Dirac brackets for independent canonical variables. Instead we suggest the usual Poisson bracket relations

$$\{A_{ia}(x), \pi_b^j(y)\} = \delta_i^j \delta_{ab} \delta^3(x - y). \quad (11)$$

We then introduce Hilbert space in which the variables  $\pi_a^i(x)$  act as differential operators  $-i\delta/\delta A_{ia}(x)$  and consider the constraints as the operator conditions:

$$\Phi_a(x)\Psi = D_i \pi_a^i(x)\Psi = 0, \quad (12)$$

in which  $\Psi$  are the vectors of Hilbert space and obey the Schrödinger equation.

$$H\Psi = E\Psi. \quad (13)$$

In what follows we need the expressions of conserved physical quantities as a result of the invariance of the equations of motion with respect to the translations and rotations in 3-dimensional Euclidean space or the transformations which constitute the group  $M(3)$  of the motion of the space. According to the Noether's theorem one obtains the following expressions for the components of the momentum and angular momentum vectors of the system:

$$P^i = - \int d^3x \frac{\partial A_{ja}}{\partial x_i} \pi_a^j, \quad (14)$$

$$M_i = \int d^3x \{ -(I_i)_k{}^j x_j \frac{\partial A_{na}}{\partial x_k} \pi_a^n + (I_i)_k{}^j A_{ja} \pi_a^k(x) \}, \quad (15)$$

in which  $(I_i)_k{}^j = \varepsilon_{ik}{}^j$  are the elements of the adjoint representation of  $SO(3)$ . We construct below the perturbation theory around the localized finite energy configuration which takes into consideration the conservations laws of momentum and angular momentum exactly in every order of perturbation theory.

### 3. Collective coordinates

The solutions of the equations of motion can be considered as the functions of certain numbers of parameters, which is the result of invariance

of the equations of motion with respect to the symmetry transformations. In accordance with this we introduce the following transformation of fields  $A_{ia}(x)$ <sup>2</sup>:

$$A_{ia}(x) = C_i{}^j(\theta) \left[ \frac{1}{g} w_{ja}(\overline{C}(\mathbf{x} - \mathbf{q})) + W_{ja}(\overline{C}(\mathbf{x} - \mathbf{q})) \right]. \quad (16)$$

We introduce with this transformation six parameters (three rotational  $\theta_j$  and three translational  $q_j$ ,  $j = 1, 2, 3$ ) as new variables which constitute together with  $W_{ja}(\mathbf{x})$  and the conjugate momenta of the latter the enlarged phase space. In order to retain the original number of independent variables we must subject them to as many additional conditions as the number of introduced parameters. We choose the following linear conditions:

$$\int d^3x N_{Ik,b}{}^j(\mathbf{x}) W_{jb}(\mathbf{x}) = 0, \quad I = 1, 2, \quad k = 1, 2, 3, \quad (17)$$

with  $c$ -number functions  $N_{Ik,b}{}^j(\mathbf{x})$ . One finds always the another set of  $c$ -number functions  $M_{jb,Jk}(\mathbf{x})$ , such that the following orthonormality conditions hold:

$$\int d^3x N_{Ik,b}{}^j(\mathbf{x}) M_{jb,Jn}(\mathbf{x}) = \delta_{IJ} \delta_{kn}. \quad (18)$$

We calculate the variations  $\delta/\delta A_{ia}(x)$  in the usual way:

$$\frac{\delta}{\delta A_{ia}(x)} = \frac{\delta q_j}{\delta A_{ia}(x)} \frac{\partial}{\partial q_j} + \frac{\delta \theta_j}{\delta A_{ia}(x)} \frac{\partial}{\partial \theta_j} + \int d^3y \frac{\delta W_{jb}(y)}{\delta A_{ia}(x)} \frac{\delta}{\delta W_{jb}(y)}. \quad (19)$$

Determining  $W_{jb}(y)$  from (16) and substituting into (17) we then calculate the variation of (17) with respect to  $A_{ia}(\mathbf{x})$ . One obtains in this way the system of algebraic equations :

$$\overline{C}_j{}^i N_{Ik,a}{}^j(\overline{C}(\mathbf{x} - \mathbf{q})) + \frac{1}{g} \tilde{N}_{Ik,a}{}^i(\mathbf{x}) + F_{Ik,J}{}^s \tilde{N}_{Js,a}{}^i(\mathbf{x}) = 0 \quad (20)$$

with the functions  $\tilde{N}_{Js,a}{}^i(\mathbf{x})$  as unknowns. The relations of the functions  $\tilde{N}_{Js,a}{}^i(\mathbf{x})$  to the quantities we are interested in are determined by the equalities (see the Appendix for some notations ):

$$\frac{\delta \theta_m}{\delta A_{ia}(\mathbf{x})} = \overline{A}_m{}^k(\theta) \tilde{N}_{1k,a}{}^i(\mathbf{x}), \quad \frac{\delta q_m}{\delta A_{ia}(\mathbf{x})} = \overline{C}_m{}^k \tilde{N}_{2k,a}{}^i(\mathbf{x}).$$

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<sup>2</sup>The matrix  $C$  is a matrix of 3-dimensional rotations. The inverse matrix to  $C$  is denoted by  $\overline{C}$ .

The quantities  $F_{Ik,Js}$  are

$$F_{Ik,1s} = \int d^3x N_{Ik,b}{}^j(\mathbf{x}) [-(\bar{I}_s)_m{}^n x_n \frac{\partial W_{jb}(\mathbf{x})}{\partial x_m} + (\bar{I}_s)_j{}^n W_{nb}(\mathbf{x})], \quad (21)$$

$$F_{Ik,2s} = \int d^3x N_{Ik,b}{}^j(\mathbf{x}) \frac{\partial W_{jb}(\mathbf{x})}{\partial x^s}. \quad (22)$$

Thus the quantities  $\delta q_j/\delta A_{ia}(x)$ ,  $\delta \theta_j/\delta A_{ia}(x)$  will be determined with the use of the system of algebraic equations and therewith the quantities  $M_{ia,Ik}(\mathbf{x})$  are determined, namely

$$M_{1s,jb}(\mathbf{x}) = (-\bar{I}_s)_l{}^r x_r \frac{\partial w_{jb}(\mathbf{x})}{\partial x_l} + (\bar{I}_s)_j{}^l w_{lb}(\mathbf{x}), \quad M_{2s,jb}(\mathbf{x}) = \frac{\partial w_{jb}(\mathbf{x})}{\partial x^s}.$$

In calculating  $\delta W_{jb}(y)/\delta A_{ia}(x)$  we have to take into account the additional conditions (17). This can be done by defining a projection matrix

$$P_{ia,jb}(\mathbf{x}, \mathbf{y}) = \delta_{ij} \delta_{ab} \delta^3(\mathbf{x} - \mathbf{y}) - M_{ia,Ik}(\mathbf{x}) N_{I,jb}^k(\mathbf{y}), \quad (23)$$

which has the following properties:

$$\int d^3x N_{Ik,b}{}^j(\mathbf{x}) P_{jb,ia}(\mathbf{x}, \mathbf{y}) = \int d^3y P_{ia,jb}(\mathbf{x}, \mathbf{y}) M_{b,Ik}^j(\mathbf{y}) = 0, \quad (24)$$

$$\int d^3y P_{ia,jb}(\mathbf{x}, \mathbf{y}) P_{b,kc}^j(\mathbf{y}, \mathbf{z}) = P_{ia,kc}(\mathbf{x}, \mathbf{z}). \quad (25)$$

This operator makes it possible to determine the variables  $W_{ia}(\mathbf{x})$  (subjected to (17)) as the linear combination of independent variables -  $W_{ia}(\mathbf{x}) = \int d^3y P_{ia,jb}(\mathbf{x}, \mathbf{y}) V_b^j(\mathbf{y})$  with  $V_b^j(\mathbf{x})$  being expressed through  $A_{ia}(\mathbf{y})$ ,  $q_j$  and  $\theta_j$  with the help of (16). One obtains in this way:

$$\begin{aligned} \frac{\delta W_{kb}(\mathbf{y})}{\delta A_{ia}(\mathbf{x})} &= P_{kb,ja}(\mathbf{y}, \bar{C}(\mathbf{x} - \mathbf{q})) \bar{C}^{ji} + \\ &+ \tilde{N}_{1s,a}{}^i(\mathbf{x}) \int d^3z P_{kb,c}{}^n(\mathbf{y}, \mathbf{z}) [-(\bar{I}^s)_l{}^r z_r \frac{\partial W_{nc}(\mathbf{z})}{\partial z_l} + (\bar{I}^s)_n{}^j W_{jc}(\mathbf{z})] + \\ &+ \tilde{N}_{2s,a}{}^i(\mathbf{x}) \int d^3z P_{kb,c}{}^j(\mathbf{y}, \mathbf{z}) \frac{\partial W_{jc}(\mathbf{z})}{\partial z_s}. \end{aligned} \quad (26)$$

We give now the final expression for the momenta  $\pi_a^i(\mathbf{x})$ :

$$\begin{aligned} \pi_a^i(\mathbf{x}) &= \bar{C}_j{}^i \{ \Pi_a^j(\bar{C}(\mathbf{x} - \mathbf{q})) - \\ &- g[\mathbb{I} + gF]^{-1sk}{}_{IJ} N_{Jk,a}{}^j(\bar{C}(\mathbf{x} - \mathbf{q})) (\bar{l}_{Is} + M_{Is}(\Pi)) \}. \end{aligned} \quad (27)$$

The quantities  $\Pi_a^j(\mathbf{x})$  are momenta obtained by projecting the functional derivatives  $-i\delta/\delta W_{ia}(\mathbf{x})$ ,

$$\Pi_a^i(\mathbf{x}) = \int d^3y P_{jb, a}^i(\mathbf{y}, \mathbf{x}) \left( -i \frac{\delta}{\delta W_{jb}(\mathbf{x})} \right),$$

and has the following commutation relations with the  $W_{ia}(\mathbf{x})$ :

$$[W_{ia}(\mathbf{x}), \Pi_b^j(\mathbf{y})] = iP_{ia, b}^j(\mathbf{x}, \mathbf{y}). \quad (28)$$

Besides they satisfy the constraint equation:

$$\int d^3x M_{ia, Ik}(\mathbf{x}) \Pi_a^i(\mathbf{x}) = 0 \quad (29)$$

The dependence of momenta  $\pi_a^i(\mathbf{x})$  on the variables  $q_i$ ,  $\theta_i$  are defined by the operators  $\bar{l}_{Is}$ . They are generators of inverse transformations from the group  $M(3)$  (see appendix), the first three of which -  $\bar{l}_s$  generate the inverse rotations and others -  $\bar{C}_{ij}\partial/i\partial q_j$  translations in the opposite direction (we define as forward transformations  $\mathbf{x} \rightarrow \bar{C}\mathbf{x} + \mathbf{q}$ ). It is significant at this point to have the conserved quantities  $P^i$  and  $M_i$  expressed through the new set of phase space variables. One can show, that

$$P_i = l_{1i} = -i \frac{\partial}{\partial q_i}, \quad M_i = l_{2i} = l_i - i\varepsilon_{ijk} q_j \frac{\partial}{\partial q_k}, \quad (30)$$

which coincide with six generators  $l_{Is} = (l_{1s}, l_{2s})$  of forward transformations of the group  $M(3)$ . The generators  $l_i$  are generators of  $SO(3)$ . These relations together with the expression (27) are important proceeding from the statement that the generators of forward transformations commute with those of backward transformations:

$$[l_{Ii}, \bar{l}_{J,k}] = 0. \quad (31)$$

This means that the dependence of the state vector  $\Psi$  on variables  $q_i$ ,  $\theta_i$  can be factorized and the operators  $l_{Ik}$  can be replaced by c-number quantities. Before doing this we need to perform a transformation of the state vector  $\Psi$ , namely take

$$\Psi = \exp \left( \frac{i}{g} \int d^3x s_a^i(\mathbf{x}) W_{ia}(\mathbf{x}) \right) \Psi', \quad (32)$$



which means that the momenta  $\Pi_a^i(\mathbf{x})$  have to be replaced by  $\frac{1}{g}s_a^i(\mathbf{x}) + \Pi_a^i(\mathbf{x})$ . The quantities  $s_a^i(\mathbf{x})$  are subject to the same additional conditions as  $\Pi_a^i(\mathbf{x})$ :

$$\int d^3x M_{ia,Ik}(\mathbf{x}) s_a^i(\mathbf{x}) = 0. \quad (33)$$

These conditions can always be fulfilled. If it is not so, then one defines with the help of a projection operator a new quantity  $s_{ia}'(\mathbf{x})$  which satisfies the needed conditions. After performing this transformation one obtains for  $\pi_a^i(\mathbf{x})$ :

$$\begin{aligned} \pi_a^i(\mathbf{x}) = & \bar{C}_j^i \left\{ \frac{1}{g} s_a^j(\bar{C}(\mathbf{x} - \mathbf{q})) + \Pi_a^j(\bar{C}(\mathbf{x} - \mathbf{q})) - \right. \\ & \left. -g[\mathbb{I} + gF]^{-1sk}_{IJ} N_{Jk,a}^j(\bar{C}(\mathbf{x} - \mathbf{q})) (\bar{l}_{Is} + \frac{1}{g} M_{Is}(s) + M_{Is}(\Pi)) \right\}. \end{aligned} \quad (34)$$

This completes the preparation of all formulae which are needed to construct the perturbation theory.

#### 4. Quantum behaviour of localized solutions

In this section we consider the Schrödinger equation

$$H\Psi' = E\Psi' \quad (35)$$

which is supplemented by the additional conditions:

$$D_i \pi_a^i(x) \Psi' = 0, \quad (36)$$

with  $\pi_a^i(x)$  as in Eq. (34). Writing out explicitly the kinetic energy we see, that dependence of the Hamiltonian on the group parameters is determined by generators  $\bar{l}_{Ik}$ . This fact enables us to separate the dependence of the state vector on the parameters of the symmetry group  $M(3)$  in the form of a factor as an indication of the exact fulfillment of the conservation laws, i.e.

$$\Psi' = T(q_i, \theta_i) \Psi''((W_{ia}(\mathbf{x}), \Pi_a^i(\mathbf{x})) \quad (37)$$

(in which  $T(q_i, \theta_i)$  realizes the representation of the group  $M(3)$  [14]) and to replace the generators  $\bar{l}_{Ik}$  by appropriate  $c$ -numbers:  $\bar{l}_{1k} \rightarrow \bar{J}_k$ ,  $\bar{l}_{2i} \rightarrow (1/g^2)K_i$  with the perturbation order of the momentum being raised, which

makes it possible to see the translational effects already in the first approximation.

The eigenvalues of the Schrödinger equation calculated below perturbatively are rigorously consistent with the symmetry properties of the physical system. Thus any approach for solving the Schrödinger equation describes the symmetry properties exactly in every order of approximation.

We can now expand the Hamiltonian and the constraint as a series in inverse powers of  $g$ . The energy and state vector also must be expanded in appropriate series. Thus we have:

$$H = g^2 H_0 + g H_1 + H_2 + g^{-1} H_3 + g^{-2} H_4 + \dots, \quad (38)$$

$$E = g^2 E_0 + g E_1 + E_2 + g^{-1} E_3 + g^{-2} E_4 + \dots, \quad (39)$$

$$\Psi'' = \Psi_0 + g^{-1} \Psi_1 + \dots. \quad (40)$$

The leading (or zero) approximation of the perturbation theory

$$(H_0 - E_0)\Psi_0 = 0$$

does not consist of field operators and is completely determined by  $c$ -numbers, so the energy of zeroth order is

$$E_0 = \int d^3x \left\{ \frac{1}{4} f_{ija} f_{ija} + \frac{1}{2} (s_{ia}(\mathbf{x}) - N_{2j,ia}(\mathbf{x}) K^j) (s_{ia}(\mathbf{x}) - N_{2j,ia}(\mathbf{x}) K^j) \right\} \quad (41)$$

with arbitrary nonzero  $\Psi_0$ . The numbers  $s_{ia}(\mathbf{x})$  remain in this order undetermined. The quantity  $f_{ija}$  is a strength tensor, constructed by means of the classical field solutions. The next approximation reads:

$$(H_1 - E_1)\Psi_0 = 0. \quad (42)$$

The operator  $H_1$  is linear in  $W_{ia}(\mathbf{x})$ ,  $\Pi_a^i(\mathbf{x})$ . The regularity of the function  $\Psi''$  with respect to  $W_{ia}(\mathbf{x})$ ,  $\Pi_a^i(\mathbf{x})$  requires  $H_1$  and  $E_1$  to be zero identically. This leads us to the determination of the quantities  $s_{ia}(\mathbf{x})$ , namely if we choose

$$s_{ia}(\mathbf{x}) - N_{2j,ia}(\mathbf{x}) K^j = M_{ia,2k}(\mathbf{x}) c^k, \quad (43)$$

then the terms linear in  $\Pi_a^i(\mathbf{x})$  vanish. Collecting the terms at  $W_{ia}(\mathbf{x})$  we obtain:

$$H_1 = \int d^3x \left[ c_m c_n \frac{\partial^2 w_{jc}(\mathbf{x})}{\partial x_m \partial x_n} - \frac{\partial f_{ijc}}{\partial x_i} - \varepsilon_{cba} w_b^i(\mathbf{x}) f_{ija} \right] W_c^j(\mathbf{x}) = 0. \quad (44)$$

We see that the terms in front of  $W_{jc}(\mathbf{x})$  coincide with the classical Euler-Lagrange equations provided  $w_{0a}(\mathbf{x}) = 0$ . Thus  $H_1$  is identically zero. On substituting (43) into the expression for  $E_0$  we are reassured that this is just the energy of the classical localized configurations, provided  $w_{0a}(\mathbf{x}) = 0$ . We return now to the Eq. (33). On the basis of (43) it is easy to obtain from (33) the following expression for  $K^j$ :

$$K^j = c_i \int d^3x \frac{\partial w_{na}(\mathbf{x})}{\partial x_i} \frac{\partial w_a^n(\mathbf{x})}{\partial x_j} \quad (45)$$

which are just components of the classical momentum. Considering now the derivative of the energy  $E_0$  with respect to  $c_k$ , we find that:

$$c_i = g^2 \frac{\partial E_0}{\partial K^i}. \quad (46)$$

This means, that  $\mathbf{c}$  is a velocity of the center-of-mass of the system.

At this level the wave function  $\Psi_0(W_{jc}(\mathbf{x}), \Pi_a^i(\mathbf{x}))$  is still undetermined. The next approximation to the ground level energy  $E_0$  is determined from the terms of order  $g^0$ :

$$(H_2 - E_2)\Psi_0(W_{jc}(\mathbf{x}), \Pi_a^i(\mathbf{x})) = 0. \quad (47)$$

The quantity  $H_2$  is a quadratic form of the operators  $W_{ia}(\mathbf{x})$  and  $\Pi_a^i(\mathbf{x})$  and can be diagonalized being reduced to an infinite set of the oscillators. First we consider the constraints. From now on we can choose the system of center-of-mass by fixing  $\mathbf{c} = 0$ , that is  $\mathbf{K} = 0$ . This suggestion simplifies the Hamilton operator and the constraints (12). In this system we obtain for  $H_2$ :

$$H_2 = \int d^3x \left[ \frac{1}{2} \Pi_{ai}(\mathbf{x}) \Pi_{ai}(\mathbf{x}) + W_{ib}(\mathbf{x}) \frac{1}{2} \hat{O}_{ib,jc}(\mathbf{x}) W_{jc}(\mathbf{x}) \right], \quad (48)$$

in which

$$\hat{O}_{ib,jc} = -\delta_{ij}(D_k^{cl} D^{clk})_{bc} + (D_i^{cl} D_j^{cl})_{bc} + 2\varepsilon_{abc} f_{ija} \quad (49)$$

with

$$(D^{cl}_j)_{ac} = \delta_{ac} \frac{\partial}{\partial x^j} + \varepsilon_{abc} w_{jb}(\mathbf{x}). \quad (50)$$

The operator  $\hat{O}_{ib,jc}$  depends on the classical configurations and, of course, the knowledge of the exact form of such solutions is needed in order to describe the corresponding spectrum of energy. We are interested mostly in that part of the spectrum which arises due to the symmetry properties of the system and therefore we perform the diagonalization of  $H_2$  only formally. Suppose the functions  $U_{ia}^{(bj)}(\mathbf{x}, n)$  constitute a set of the solutions of the following system of equations:

$$\hat{O}_{ib,c}{}^j U_{jc}(\mathbf{x}, n) = E(n) U_{ib}(\mathbf{x}, n) \quad (51)$$

with the orthonormality conditions:

$$\int d^3x U_{ia}^{*(bj)}(\mathbf{x}, n) U_{ia}^{(b'j')}(\mathbf{x}, n') = \delta_{bb'} \delta_{jj'} \delta_{nn'} \quad (52)$$

We now consider the expansion of the field  $W_{ia}(\mathbf{x})$  in terms of  $U_{ia}^{(bj)}(\mathbf{x}, n)$ :

$$W_{ia}(\mathbf{x}) = \sum_n' \sqrt{\frac{1}{2E(n)}} \left( \alpha_{bk}(n) e_j^{(k)} U_{ia}^{(bj)}(\mathbf{x}, n) + \alpha_{bk}^+(n) e_j^{(k)} U_{ia}^{*(bj)}(\mathbf{x}, n) \right), \quad (53)$$

where the prime indicates that the sum does not contain the modes with zero energy (related to the translations and rotations). The summation number  $n$  may in general be composed of discrete and continuous variables. The creation and annihilation operators  $\alpha_{bj}(n)$ ,  $\alpha_{bj}^+(n)$  obey the commutation relations

$$[\alpha_{ia}(n), \alpha_{bj}^+(n')] = \delta_{ij} \delta_{ab} \delta_{nn'}. \quad (54)$$

The polarization vectors  $e_j^{(k)}$  make an orthonormal set of unit vectors in 3-dimensional Euclidean space. We next have to deal with the unphysical degrees of freedom related to the constraints. Up to order  $g^0$  the constraints are:

$$(D^{cl}_j)_{ac} \Pi_c^j(\mathbf{x}) \Psi'' = 0, \quad (55)$$

which are linear in field operators. The unphysical degrees of freedom related to these constraints can be eliminated by introducing the Coulomb gauge conditions written as expectation value

$$(\Psi'', \partial^i W_{ia}(\mathbf{x}) \Psi'') = 0. \quad (56)$$

Since (55) and (56) are linear, we can consider the state vector obtained by acting on the vacuum with the creation operators pertaining to the various polarizations, so the state vector might be represented as

$$\Psi'' = \Psi^t \Psi^l, \quad (57)$$

where  $\Psi^t$  corresponds to the transverse components of the field  $W_{ia}(\mathbf{x})$ , while  $\Psi^l$  is obtained by acting on the vacuum with longitudinal creation operators. We demand that the positive frequency part of  $\partial^i W_{ia}(\mathbf{x})$  annihilates the vector of Hilbert space. We are interested in examining the consequences of gauge conditions on the state  $\Psi^l$ . So we may write

$$\sum_n \alpha_{bk}(n) e_j^{(k)} A_{ba}{}^j \Psi^l = 0, \quad (58)$$

with  $A_{ab}{}^j = \partial^i U_{iba}{}^j(\mathbf{x}, n)$ . Since we choose the polarization vectors orthogonal to the quantities  $A_{ab}{}^j$ , (58) can be written in the equivalent form:

$$\alpha_{3a} \Psi^l = 0 \quad (59)$$

We present the state  $\Psi^l$  as the linear combination of the eigenstates  $|n\rangle$  of the quantum number operator  $N = \sum_n \alpha_{3a}^+(n) \alpha_{3a}(n)$  corresponding to the longitudinal degrees of freedom. The states  $|n\rangle$  obeying (59) satisfy the condition

$$n < n | n \rangle = 0 \quad \text{or} \quad \langle n | n \rangle = \delta_{n0}$$

and therefore for the state  $\Psi^l$  as linear combination of  $|n\rangle$  the following relation holds:  $(\Psi^l, \Psi^l) = c < 0 | 0 \rangle \neq 0$ . Expanding the operators in a series similar to (53) we have

$$\Pi_{ia}(\mathbf{x}) = i \sum_n' \sqrt{\frac{E(n)}{2}} \left( \alpha_{bk}^+(n) e_j^{(k)} U_{ia}^{*(bj)}(\mathbf{x}, n) - \alpha_{bk}(n) e_j^{(k)} U_{ia}^{(bj)}(\mathbf{x}, n) \right), \quad (60)$$

and substituting (53), (60) into (48) we obtain:

$$H_2 = \frac{1}{2} \sum_n \left( \alpha_{bk}^+(n) \alpha_{bk}(n) + \alpha_{bk}(n) \alpha_{bk}^+(n) \right), \quad (61)$$

which has to be understood as the normal product. Calculating the expectation value of  $H_2$  we see that only transversal components contribute to

the energy, since the action of  $H_2$  on the state (57) cancels the longitudinal contribution.

In order to investigate the internal structure of the system associated with spin, we must deal with the Hamiltonian up to the order  $g^2$ . Combining the terms of order  $g$ , that is  $H_3$ , we can show that after averaging over the ground-state wave function  $\Psi''$  its contribution to the energy is zero. We do not give the explicit form of  $H_4$  here since this consists of a lot of terms. We just indicate that it is a quartic in field operators.

The expectation value of  $H_4$  gives the contribution to the energy, which results in a splitting of the ground-state energy. Omitting the irrelevant for the structure of the energy spectrum additive terms we obtain for the energy above the ground level:

$$E_4 = \frac{1}{2} \left( \int d^3x N_{ia,1k}(\mathbf{x}) N_{ia,1,k}(\mathbf{x}) \right) j(j+1) \quad (62)$$

which describes the excited quantum states with the spin  $j$ . The number  $j$  takes either integer or half-integer values. Since the initial vector fields  $A_{\mu a}$  belong to the adjoint representation of the group of rotations  $j$  must be integer, hence  $j = 0, 1, 2, \dots$ . Thus, we see that there are excitation modes in quantum theory of the localized finite energy solutions which are labelled by the quantum numbers corresponding to the spin of the system. Besides there is a degeneracy of the energy spectrum with respect to the azimuthal quantum number. So the energy  $E_4$  defines a spin content of the energy spectrum.

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## Appendix

We recall here some relevant formulae from group theory. Let the matrices  $C$  be the elements of the group  $M$ . We denote by  $\overline{C}$  the inverse transformations. If the parameters of the group are  $\theta_i$  then the following equations hold:

$$\frac{\partial C}{\partial \theta_i} = B^{ij}(\theta) T_j C,$$

in which  $T_j$  are the elements of the corresponding algebra. The matrix  $B^{ij}(\theta)$  is invertible -  $A_{ik}(\theta)B^{kj}(\theta) = \delta_i^j$ . With the matrix  $A_i{}^k(\theta)$  the generators of the group are determined:

$$L_i = A_{ik}(\theta) \frac{\partial}{\partial \theta_k}$$

with the commutation relations

$$[L_i, L_k] = f_{ikj} L^j.$$

The inverse matrices obey the same equations, but with quantities being overlined:

$$\frac{\partial \bar{C}}{\partial \theta_i} = \bar{B}^{ij}(\theta)(\bar{T}_j)\bar{C}, \quad \bar{L}_i = \bar{A}_{ik}(\theta) \frac{\partial}{\partial \theta_k}, \quad [\bar{L}_i, \bar{L}_k] = f_{ikj} \bar{L}^j,$$

so that  $\bar{L}_i = C_i^j L_j$ ,  $\bar{T}_i = -T_i$  and  $[\bar{L}_i, L_j] = 0$ . The generators of the forward transformations of the group  $M(3)$  we are interested in are

$$l_{1i} = l_i + \varepsilon_{ijk} q^j \frac{\partial}{i \partial q_k}, \quad l_{2i} = \frac{\partial}{i \partial q^i}.$$

and the inverse transformations are

$$\bar{l}_{1i} = \bar{l}_i, \quad \bar{l}_{2i} = \bar{C}_{ij} \frac{\partial}{i \partial q_j}.$$

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